

Curves on Flat Tori and Analog Source-Channel Codes

Antonio Campello, *Student Member, IEEE*, Cristiano Torezzan, and Sueli I. R. Costa, *Member, IEEE*

Abstract

In this paper we consider the problem of transmitting a continuous alphabet discrete-time source over an AWGN channel. We propose a constructive scheme based on a set of curves on the surface of a $2N$ -dimensional sphere. Our approach shows that the design of good codes for this communication problem relies on geometrical properties of spherical codes and projections of N -dimensional rectangular lattices. Theoretical comparisons with some previous works in terms of the mean square error as a function of the channel SNR as well as simulations are provided.

I. INTRODUCTION

The problem of designing good codes for continuous alphabet sources to be transmitted over a channel with power constraint can be viewed as the one of constructing curves in the Euclidean space of maximal length and such that its folds (or laps) are a good distance apart. When the channel noise is below a certain threshold, a bigger length essentially means a higher resolution when estimating the sent value. On the other hand, if the folds of the curve come too close, this threshold will be small and the mean square error (mse) will be dominated by larger errors as consequence of decoding to the wrong fold. Explicit constructions of curves for analog source-channel coding were presented, for example, in [9] and [10].

In this work, we will consider spherical curves in the $2N$ -dimensional Euclidean space. We develop an extension of the construction presented in [9] to a set of curves on layers of flat tori. Our approach explores the geometrical properties of spherical codes and projections of N -dimensional lattices. In the scheme presented here, homogeneity and low decoding complexity properties were preserved whereas the total length can be meaningfully increased. Theoretical results as well as simulations show that this approach outperforms previous ones in terms of the tradeoff between SNR and mean square error.

This paper is organized as follows: in Sections II and III, we introduce some mathematical/information-theoretical tools used in our approach. In Section IV we present a scheme to design piecewise homogeneous curves on the Euclidean sphere and describe the encoding/decoding process. In Section V we derive an extended version of the

A. Campello and S. Costa are with Institute of Mathematics, Statistics and Computer Science, University of Campinas, São Paulo, Brazil, 13083-759

C. Torezzan is with School of Applied Sciences, University of Campinas, São Paulo, Brazil, 13484-350

Part of this work was presented in the International Symposium of Information Theory (ISIT) 2012, Boston - MA.

Lifting Construction [7] suitable to our problem and present some examples and length comparisons with some previous constructions. In the last sections we present theoretical and experimental analysis of the proposed scheme.

II. COMMUNICATION FRAMEWORK

The communication system we considered here is illustrated below. The objective is to transmit an input real value x over a channel of dimension N with additive Gaussian noise and average power constraint. The encoder will use a bandwidth expansion mapping in order to encode x to a point $\mathbf{s}(x) \in \mathbb{R}^N$ respecting the constraint $E[\mathbf{s}(x)\mathbf{s}(x)^t] \leq P$. The decoder will recover an estimate \hat{x} for the sent value attempting to minimize the mean square error (mse) $E[(X - \hat{X})^2]$. The source will be assumed to have pdf with support $[0, 1]$ and $\mathbf{s}([0, 1])$ will be referred as the *signal locus*. If \mathbf{s} is a continuous mapping, then the signal locus is a curve in \mathbb{R}^N .

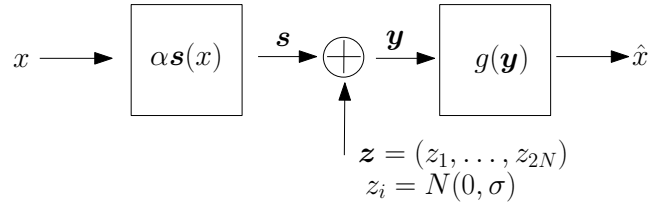


Fig. 1: Communication system

This communication model is carefully analyzed in [6, Ch. 4]. Given a radius $\rho > 0$, let E_ρ denote the event $\|\mathbf{z}\| < \rho$. The mse can be split into two terms such that:

$$\text{mse} = E[(X - \hat{X})^2 | E_\rho] P(E_\rho) + E[(X - \hat{X})^2 | E_\rho^c] P(E_\rho^c). \quad (1)$$

For high values of the SNR P/σ^2 , $P(E_\rho^c) \approx 0$ and the mse is well approximated by $E[(X - \hat{X})^2 | E_\rho]$ (the goodness of the approximation depends on the SNR and ρ). In this case, called the *low noise regime*, it is possible to show [6] that:

$$E[(X - \hat{X})^2] \approx \sigma^2 \int_0^1 f(x) \|\dot{\mathbf{s}}(x)\|^{-2} dx := E_{\text{low}}[(X - \hat{X})^2], \quad (2)$$

where $f(x)$ is the pdf of the source. Most of the analysis in Section VI will be done under the low noise regime. If the source is uniform, then an optimal parametrization for $\mathbf{s}(x)$ yields

$$E_{\text{low}}[(X - \hat{X})^2] = \frac{\sigma^2}{L^2}, \quad (3)$$

where L is the curve length.

A geometric picture of the possible effects of the noise on the estimated value is given by Figure 1. Under the low noise regime, all the calculations can be essentially done by approximating $\mathbf{s}(x)$ for its tangent line and the system will be described by Equation (2). However, this is not true for large errors. Decoding to the wrong fold of

the curve yields a poor performance of the system. This suggests that our objective is to choose long curves with a large distance between its folds.

In our constructions, the signal locus will be a collection of curves in \mathbb{R}^N rather than a single curve (this can be viewed as a “piecewise continuous” curve). Our analysis is mainly focused on uniform sources, although the scheme can be easily adapted to any source with limited support and even to Gaussian sources, by means of a companding function [6].

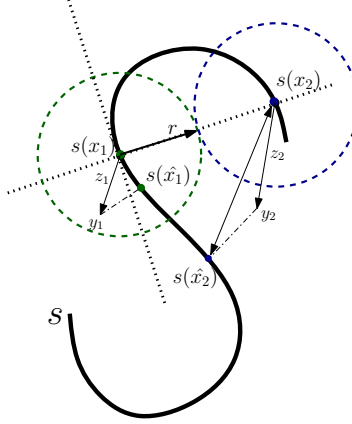


Fig. 2: Small errors and large errors.

III. BACKGROUND

A. Flat Tori

The unit sphere $S^{2N-1} \subset \mathbb{R}^{2N}$ can be foliated with flat tori [1], [8] as follows. For each unit vector $\mathbf{c} = (c_1, c_2, \dots, c_N) \in \mathbb{R}^N$, $c_i > 0$, and $\mathbf{u} = (u_1, u_2, \dots, u_N) \in \mathbb{R}^N$, let $\Phi_{\mathbf{c}} : \mathbb{R}^N \rightarrow \mathbb{R}^{2N}$ be defined by

$$\Phi_{\mathbf{c}}(\mathbf{u}) = \left(c_1 \left(\cos \frac{u_1}{c_1}, \sin \frac{u_1}{c_1} \right), \dots, c_N \left(\cos \frac{u_N}{c_N}, \sin \frac{u_N}{c_N} \right) \right). \quad (4)$$

This N -periodic function $\Phi_{\mathbf{c}}$ is a local isometry on its image, the torus $T_{\mathbf{c}}$, a flat N -dimensional surface contained in the unit sphere S^{2N-1} . $T_{\mathbf{c}} = \Phi_{\mathbf{c}}(\mathbb{R}^N)$ is also the image of the hyperbox:

$$\mathcal{P}_{\mathbf{c}} := \{\mathbf{u} \in \mathbb{R}^N; 0 \leq u_i < 2\pi c_i\}, \quad 1 \leq i \leq N. \quad (5)$$

Note also that each vector of S^{2N-1} belongs to one, and only one, of these flat tori if we also consider the degenerated cases where some c_i may vanish. Thus we say that the family of flat tori $T_{\mathbf{c}}$ and their degenerations, with $\mathbf{c} = (c_1, c_2, \dots, c_N)$, $\|\mathbf{c}\| = 1$, $c_i \geq 0$, defined above is a foliation on the unit sphere of $S^{2N-1} \subset \mathbb{R}^{2N}$.

It can be shown (Proposition 1 in [8]) that the minimum distance between two points, one in each flat torus $T_{\mathbf{b}}$ and $T_{\mathbf{c}}$, is

$$d(T_{\mathbf{c}}, T_{\mathbf{b}}) = \|\mathbf{c} - \mathbf{b}\| = \left(\sum_{i=1}^N (c_i - b_i)^2 \right)^{1/2}. \quad (6)$$

The distance between two points on the same torus $T_{\mathbf{c}}$ given by

$$d(\Phi_{\mathbf{c}}(\mathbf{u}), \Phi_{\mathbf{c}}(\mathbf{v})) = 2\sqrt{\sum c_i^2 \sin^2(\frac{u_i - v_i}{2c_i})}$$

is bounded in terms of $\|\mathbf{u} - \mathbf{v}\|$ by the following proposition.

Proposition 1. [8] Let $\mathbf{c} = (c_1, c_2, \dots, c_N)$, $\|\mathbf{c}\| = 1$, and let $\mathbf{u}, \mathbf{v} \in \mathcal{P}_{\mathbf{c}}$. Let $\Delta = \|\mathbf{u} - \mathbf{v}\|$ and $\delta = \|\Phi_{\mathbf{c}}(\mathbf{u}) - \Phi_{\mathbf{c}}(\mathbf{v})\|$.

Then

$$\frac{2}{\pi}\Delta \leq \frac{\sin \frac{\Delta}{2c_{\xi}}}{\frac{\Delta}{2c_{\xi}}}\Delta \leq \delta \leq \frac{\sin \frac{\Delta}{2}}{\frac{\Delta}{2}}\Delta \leq \Delta \quad (7)$$

where $c_{\xi} = \min c_i$.

B. Curves

As pointed out earlier in [9], some important properties of a curve from a communication point of view are its *stretch* and *small-ball radius*. Given a curve $\mathbf{s} : [a, b] \mapsto \mathbb{R}^N$, the Voronoi region $V(x)$ of a point $\mathbf{s}(x)$ is the set of all points in \mathbb{R}^N which are closer to $\mathbf{s}(x)$ than to any other point of the curve. If $H(x)$ denotes the hyperplane orthogonal to the curve at $\mathbf{s}(x)$, then the maximal *small-ball radius* of \mathbf{s} is the largest $r > 0$ such that $B_r(\mathbf{s}(x)) \cap H(x) \subset V(x)$ for all $x \in [a, b]$, where $B_r(\mathbf{s}(x))$ is the Euclidean ball of radius r centered at $\mathbf{s}(x)$. This means that the “tube” of radius r placed along the curve does not intersect itself. The *stretch* $\mathcal{S}(x)$ is the function $\|\dot{\mathbf{s}}(x)\|$ where $\dot{\mathbf{s}}(x)$ is the derivative of $\mathbf{s}(x)$. If $\|\dot{\mathbf{s}}\|$ does not depend on x (as it is the case of the curves considered here) we will refer to the stretch as simply \mathcal{S} . The length of a curve is given by $\int_a^b \mathcal{S}(x) dx$. In this paper we will be interested in curves with large length and small-ball radius.

IV. THE TORUS LAYER SCHEME

The design of those curves in this section is essentially divided in two parts. First, we choose a collection of M tori on the surface of the Euclidean sphere S^{2N-1} at least 2δ apart. The approach for doing this is via discrete spherical codes in \mathbb{R}^N . Second, we show a systematic way of constructing curves on each layer, via projection-lattices in \mathbb{R}^{N-1} . Finally, we give a description of the whole signal locus and summarize the encoding process.

For this torus layer scheme, the unit interval will be partitioned into M intervals of different length, and each of them mapped into a curve on one of the layers. It is worth noticing that, for the special case $M = 1$, if we choose the torus associated to the vector $\mathbf{c} = \frac{1}{\sqrt{N}}(1, \dots, 1)$ to encode the information, then the scheme proposed here is exactly the one analysed in [9]. As we shown next, for $M > 1$, the curves presented here outperform the ones presented in [9] (see also [7]) in terms of length and small-ball radius.

A. Torus Layers

Given a target small ball radius δ , the first step of our approach is to define a collection $T = \{T_1, T_2, \dots, T_M\}$ of flat tori on S^{2N-1} such that the minimum distance (6) between any two of them is greater than 2δ . This step is

equivalent to design a N -dimensional spherical code \mathcal{SC}_+ with minimum distance 2δ and such that its M codewords have non-negative coordinates.

Each point $\mathbf{c} \in \mathcal{SC}_+$ defines a hyperbox $\mathcal{P}_{\mathbf{c}}$ (5) and hence a flat torus $T_{\mathbf{c}}$ in the unit sphere S^{2N-1} . By (6) we can assert that the distance between two of those tori is at least 2δ .

There are several ways of constructing spherical codes that can be employed here, e.g. [3], [4] or even on layers of flat tori as introduced in [8].

We present next an example of a family of discrete spherical codes in \mathbb{R}^N that will be useful in Section VI.

Example 1. *Let*

$$\mathbf{c}(t) = \frac{(1, 1+t, 1+2t, \dots, 1+(N-1)t)}{\sqrt{\sum_{i=0}^{N-1} (1+it)^2}}, \quad t > 0 \quad (8)$$

be a non-negative N -dimensional unit vector.

The set $\mathcal{SC}_{\mathbf{c}(t)} = \{\sigma(\mathbf{c}(t)) : \sigma \in S_N\}$ of all permutations of $\mathbf{c}(t)$ defines a N -dimensional spherical code with non-negative coordinates, minimum distance equals to

$$d(t) = \frac{t\sqrt{2}}{\sqrt{\sum_{i=0}^{N-1} (1+it)^2}},$$

and cardinality $M = N!$. This set can be used in the first step of the construction described in this section.

Note that

$$\lim_{t \rightarrow 0} \mathbf{c}(t) = \frac{1}{\sqrt{N}}(1, \dots, 1) := \hat{\mathbf{e}}, \quad (9)$$

and, for each $t > 0$, the codewords of $\mathcal{SC}_{\mathbf{c}(t)}$ are equidistant from $\hat{\mathbf{e}}$, which is associated to the maximum volume torus $T_{\hat{\mathbf{e}}}$.

Moreover, for any fixed dimension N and $d_0 \in \mathbb{R}$ sufficiently small, there exists $t > 0$ such that $d(t) = d_0$. This follows by observing that the equation

$$d_0 = \frac{t\sqrt{2}}{\sqrt{\sum_{i=0}^{N-1} (1+it)^2}}$$

has a positive root for

$$0 < d_0 < \frac{2\sqrt{3}}{\sqrt{(N-1)N(2N-1)}}.$$

Although the construction above is not the best one in terms of number of points for fixed d , it has the advantage of being highly symmetric and having a closed formula for the spherical code minimum distance.

B. Curves on Each Torus

Let $T_{\mathbf{c}}$ be a torus represented by the vector $\mathbf{c} \in \mathcal{SC}_+$ as defined in the previous section. On the surface of $T_{\mathbf{c}}$ we will consider curves of the form:

$$\mathbf{s}_{T_{\mathbf{c}}}(x) = \Phi_{\mathbf{c}}(x2\pi\hat{\mathbf{u}}), \quad (10)$$

where $C = \text{diag}(c_1, \dots, c_N)$, $\hat{\mathbf{u}} = \mathbf{u}C = (c_1 u_1, \dots, c_N u_N)$, $\Phi_{\mathbf{c}}$ is given by (4) and $x \in [0, 1]$.

Provided that $\mathbf{u} \in \mathbb{Z}^N$, $\gcd(u_i) = 1$, those curves are closed (a (u_1, \dots, u_N) -type knot), and due to periodicity and local isometry properties of $\Phi_{\mathbf{c}}$ their lengths are $2\pi \|\hat{\mathbf{u}}\|$. They are also the image through $\Phi_{\mathbf{c}}$ of the intersection between the set of lines $2\pi W$, $W = \{\hat{\mathbf{u}}x + \hat{\mathbf{n}} : \hat{\mathbf{n}} = \mathbf{n}C \text{ and } \mathbf{n} \in \mathbb{Z}^N\}$, and the box $\mathcal{P}_{\mathbf{c}}$. For $c = \hat{e}$, these are exactly the curves analyzed in [9].

Let $r_{\mathbf{c}}(\mathbf{u})$ be the minimum distance between two different lines in W . We have:

$$\begin{aligned} r_{\mathbf{c}}(\mathbf{u}) &:= \min_{\hat{\mathbf{n}} \neq k\hat{\mathbf{u}}, k \in \mathbb{Z}} \min_{\hat{x}, x} \|\hat{\mathbf{u}}x - (\hat{\mathbf{u}}\hat{x} + \hat{\mathbf{n}})\| \\ &= \min_{\hat{\mathbf{n}} \neq k\hat{\mathbf{u}}, k \in \mathbb{Z}} \min_x \|\hat{\mathbf{u}}x - \hat{\mathbf{n}}\| \\ &= \min_{\hat{\mathbf{n}} \neq k\hat{\mathbf{u}}, k \in \mathbb{Z}} \|P_{\hat{\mathbf{u}}^\perp}(\hat{\mathbf{n}})\| \\ &= \min_{\hat{\mathbf{n}} \notin \hat{\mathbf{u}}^\perp} \|P_{\hat{\mathbf{u}}^\perp}(\hat{\mathbf{n}})\|, \end{aligned} \tag{11}$$

where $P_{\hat{\mathbf{u}}^\perp}(\hat{\mathbf{n}})$ denotes the orthogonal projection of $\hat{\mathbf{n}}$ onto the hyperplane $\hat{\mathbf{u}}^\perp$ which is given by the standard projection formula

$$P_{\hat{\mathbf{u}}^\perp}(\hat{\mathbf{n}}) = \hat{\mathbf{n}} \left(I_N - \frac{\hat{\mathbf{u}}\hat{\mathbf{u}}^t}{\hat{\mathbf{u}}\hat{\mathbf{u}}^t} \right). \tag{12}$$

If we consider $\Lambda_{\mathbf{c}} = c_1\mathbb{Z} \oplus \dots \oplus c_N\mathbb{Z}$, the rectangular lattice generated by matrix C , then $r_{\mathbf{c}}(\mathbf{u})$ is the length of shortest non-zero vector of the projection¹ of $\Lambda_{\mathbf{c}}$ onto $\hat{\mathbf{u}}^\perp$. Due to (7), the small-ball radius $\delta_{\mathbf{u}, \mathbf{c}}$ of $\mathbf{s}_{T_{\mathbf{c}}}$ can be bounded in terms of $r_{\mathbf{c}}(\mathbf{u})$ as follows:

$$2c_\xi \sin\left(\frac{\pi r_{\mathbf{c}}(\mathbf{u})}{2c_\xi}\right) \leq \delta_{\mathbf{u}, \mathbf{c}} \leq 2 \sin\left(\frac{\pi r_{\mathbf{c}}(\mathbf{u})}{2}\right), \tag{13}$$

where $c_\xi = \min_i c_i$ and $c_i > 0$. Thus, for small values of $\delta_{\mathbf{u}, \mathbf{c}}$, we have $\delta_{\mathbf{u}, \mathbf{c}} \approx \pi r_{\mathbf{c}}(\mathbf{u})$. Our goal is to choose \mathbf{u} in order to maximize $r_{\mathbf{c}}(\mathbf{u})$. In addition, we also want to reach a contrary objective, which is the one of maximizing the arc length $l_{\mathbf{u}, \mathbf{c}} = 2\pi \|\hat{\mathbf{u}}\|$ of $\mathbf{s}_{T_{\mathbf{c}}}$.

The proportion (density) of the volume of the “tube” of radius $\pi r_{\mathbf{c}}(\mathbf{u})$ inside $\mathcal{P}_{\mathbf{c}}$ is precisely the packing density of the lattice $P_{\hat{\mathbf{u}}^\perp}(\Lambda_{\mathbf{c}})$, the projection of $\Lambda_{\mathbf{c}}$ onto $\hat{\mathbf{u}}^\perp$, which is given by [5]:

$$\Delta(P_{\hat{\mathbf{u}}^\perp}(\Lambda_{\mathbf{c}})) = \mathcal{V}_{N-1} \left(\frac{r_{\mathbf{c}}(\mathbf{u})^{N-1} \|\hat{\mathbf{u}}\|}{2^{N-1} \prod_{i=1}^N c_i} \right) \leq \Delta_{N-1} \tag{14}$$

where Δ_{N-1} is the density of the best lattice in dimension $(N-1)$ and \mathcal{V}_{N-1} is the volume of the $(N-1)$ -dimensional unit sphere. For the case when all entries of \mathbf{c} are equal, $\Lambda_{\mathbf{c}}$ is equivalent to \mathbb{Z}^N and it was shown in [7] that we can make the above bound as tight as we want. We will show in Section V that this is also true for an

¹In general, the projection of a lattice Λ onto a subspace H is *not* a lattice unless certain special conditions are met, e.g., when H^\perp is spanned by primitive vectors of Λ [5]. This will be always the case in this paper, since $H = \hat{\mathbf{u}}^\perp$ for a primitive vector $\hat{\mathbf{u}}$.

arbitrary \mathbf{c} i.e., that projections of the rectangular lattice $\Lambda_{\mathbf{c}}$ can also yield dense lattice packings and therefore we can construct curves on the flat torus with the parameters arbitrary close to this bound.

Example 2. Let $N = 2$. Consider the local isometry

$$\Phi_{\mathbf{c}}(\mathbf{u}) = \left(c_1 \cos \frac{u_1}{c_1}, c_1 \sin \frac{u_1}{c_1}, c_2 \cos \frac{u_2}{c_2}, c_2 \sin \frac{u_2}{c_2} \right) \quad (15)$$

on the flat torus $T_{\mathbf{c}}$ and the line segment given by $\mathbf{v}(x) = x\mathbf{v} = x(2\pi u_1 c_1, 2\pi u_2 c_2)$, $u_1, u_2 \in \mathbb{Z}$ and $0 \leq x \leq 1$. The curve $\mathbf{s}_{T_{\mathbf{c}}}(x)$ will be the composition $\Phi(\mathbf{v}(x))$ and we have

$$r_{\mathbf{c}}(\mathbf{v}) \|\mathbf{v}\| = 2\pi c_1 c_2 \Rightarrow r_{\mathbf{c}}(\mathbf{v}) = \frac{c_1 c_2}{\sqrt{u_1^2 c_1^2 + u_2^2 c_2^2}} \quad (16)$$

This curve in \mathbb{R}^4 will turn around u_1 -times the circle obtained by its projection on the first two coordinates, whereas turning around u_2 -times the circle of radius c_2 given by its last two coordinates (a type (u_1, u_2) knot on the flat torus $T_{\mathbf{c}}$). In this case, it is easy to calculate $r_{\mathbf{c}}(\mathbf{u})$ since the density of the projection lattice is equal to one. In Figure 3 it is illustrated the curve $(u_1, u_2) = (4, 5)$, with $c_1 > c_2$.

C. Encoding

Let $T = \{T_1, \dots, T_M\}$ be a collection of M tori as designed in Section IV-A. For each one of these tori, let $\mathbf{s}_{T_k}(x) = \Phi(x2\pi\hat{\mathbf{u}}_k)$, $(k = 1, 2, \dots, M)$ be the curve on T_k , determined by the vector $\hat{\mathbf{u}}_k$ (10) and consider $L = \sum_{j=1}^M L_j$, where L_k is the length of \mathbf{s}_{T_k} .

Now split the unit interval $[0, 1]$ into M pieces according to the length of each curve:

$$[0, 1] = I_1 \cup I_2, \dots \cup I_M, \text{ where} \\ I_k = \left[\frac{\sum_{j=1}^{k-1} L_j}{L}, \frac{\sum_{j=1}^k L_j}{L} \right), \text{ for } k = 1, \dots, M.$$

and consider the bijective mapping

$$f_k : I_k \rightarrow [0, 1) \\ f_k(x) = \frac{x - \sum_{j=1}^{k-1} L_j / L}{l_k / L}.$$

Then the full encoding map \mathbf{s} can be defined by

$$\mathbf{s}(x) := \mathbf{s}_{T_k}(f_k(x)), \text{ if } x \in I_k. \quad (17)$$

The stretch of \mathbf{s} will be constant and equal its total length L and the small-ball radius of \mathbf{s} is the minimum small-ball radius δ of the curves \mathbf{s}_{T_k} , provided that the distance between any pair of torus in T is at least 2δ .

To encode a value x within $[0, 1]$ we apply the map (17). The signal locus will be a set of M closed curves, each one lying on a torus layer T_k and defined by a vector \mathbf{u}_k . This whole process is illustrated in Figure 3.

If the source is uniformly distributed over $[0, 1]$, the encoding process presented above is a proper one, since all subintervals will be equally stretched (see Section VI). For other applications, however, it could be worth considering another partition.

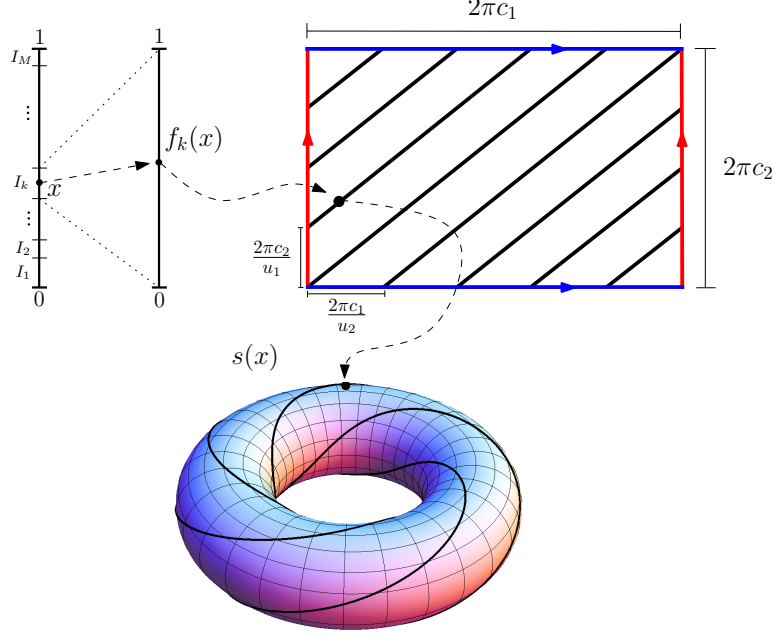


Fig. 3: Encoding Process

D. Decoding

Given a received vector $\mathbf{y} \in \mathbb{R}^{2N}$, the maximum likelihood decoding is finding \hat{x} such that:

$$\hat{x} = \arg \min_{x \in [0, 1]} \|\mathbf{y} - \mathbf{s}(x)\|.$$

Since exactly solving this problem is computationally expensive we focus on a suboptimal decoder.

For $0 \neq \gamma_i = \sqrt{y_{2i-1}^2 + y_{2i}^2}$, we can write

$$\begin{aligned} \mathbf{y} &= \left(\gamma_1 \left(\frac{y_1}{\gamma_1}, \frac{y_2}{\gamma_1} \right), \dots, \gamma_N \left(\frac{y_{2N-1}}{\gamma_N}, \frac{y_{2N}}{\gamma_N} \right) \right) \\ \mathbf{y} &= \left(\gamma_1 \left(\cos \frac{\theta_1}{\gamma_1}, \sin \frac{\theta_1}{\gamma_1} \right), \dots, \gamma_N \left(\cos \frac{\theta_N}{\gamma_N}, \sin \frac{\theta_N}{\gamma_N} \right) \right), \end{aligned}$$

where,

$$\theta_i = \arccos \left(\frac{y_{2i-1}}{\gamma_i} \right) \gamma_i, \quad 1 \leq i \leq N.$$

The process of finding the closest layer involves a N -dimensional spherical decoding of $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \dots, \gamma_N)$,

which has complexity $O(MN)$.

Let $\mathbf{c}_i = (c_{i1}, c_{i2}, \dots, c_{iN})$ be the closest point in \mathcal{SC}_+ to $\boldsymbol{\gamma}$ and

$$\bar{\mathbf{y}}_i = \left(c_{i1} \left(\cos \frac{\theta_1}{\gamma_1}, \sin \frac{\theta_1}{\gamma_1} \right), \dots, c_{iN} \left(\cos \frac{\theta_N}{\gamma_N}, \sin \frac{\theta_N}{\gamma_N} \right) \right)$$

be the projection of \mathbf{y} in the torus T_{c_i} , i.e.,

$$\|\mathbf{y} - \bar{\mathbf{y}}_i\| \leq \|\mathbf{y} - \mathbf{v}\|, \forall \mathbf{v} \in T_{c_i}.$$

From now on, we proceed the process by using a slight modification of the torus decoding algorithm [9] applied to the N -dimensional hyperbox $\mathcal{P}_{\mathbf{e}_i}$. The complexity of this algorithm is given by $O(N \|\mathbf{u}_i\|_1)$, where \mathbf{u}_i is the vector that determines the curve \mathbf{s}_{T_i} . Hence, if $M = O(\max_i \|\mathbf{u}_i\|_1)$, the overall complexity of the process described in this section will be $O(N \max_i \|\mathbf{u}_i\|_1)$, the same as for the torus decoding.

V. A SCALED LIFTING CONSTRUCTION

A. The construction

The *Lifting Construction* was proposed in [7] as a solution to the problem of finding dense lattices which are equivalent to orthogonal projections of \mathbb{Z}^N along integer vectors (“fat-strut” problem). In this section we adapt that strategy in order to construct projections of the lattice $\Lambda_{\mathbf{e}}$ which approximate any $(N-1)$ dimensional lattice (hence the densest one) with the objective of finding curves in \mathbb{R}^N approaching the bound (14). We adopt here the lattice terms as in [2]. For our purposes, the proximity measure for lattices will be the distance between their Gram matrices, as in [7]. This notion measures how close a lattice is to another one up to congruence transformations (rotations or reflections).

We consider the dual of a lattice $\Lambda \in \mathbb{R}^N$

$$\Lambda^* = \{\mathbf{x} \in \text{span}(\Lambda) : \langle \mathbf{x}, \mathbf{y} \rangle \in \mathbb{Z} \ \forall \mathbf{y} \in \Lambda\},$$

where $\text{span}(\Lambda)$ is the subspace spanned by a basis of Λ . Now let $\Lambda_{\mathbf{e}} = c_1\mathbb{Z} \oplus \dots \oplus c_N\mathbb{Z}$ be the rectangular lattice generated by the diagonal matrix C (and with Gram matrix $CC^t = C^2$). By scaling $\Lambda_{\mathbf{e}}$, we may assume that $c_1 = 1$. With this condition, if $P_{\mathbf{u}^\perp}(\Lambda_{\mathbf{e}})^*$ denotes the dual of the projection of $\Lambda_{\mathbf{e}}$ through a vector $\hat{\mathbf{u}} = (1, u_2c_2, \dots, u_Nc_N)$ ($u_i \in \mathbb{Z}$), then a generator matrix for $P_{\hat{\mathbf{u}}^\perp}(\Lambda_{\mathbf{e}})^*$ is given by

$$M = \begin{pmatrix} -u_2 & 1/c_2 & 0 & \dots & 0 \\ -u_3 & 0 & 1/c_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -u_n & 0 & 0 & \dots & 1/c_n \end{pmatrix}, \quad (18)$$

what can be derived as a consequence of Prop. 1.3.4 [5]. In what follows we derive a general construction of projections such that $P_{\mathbf{u}^\perp}$ is arbitrarily close to a target lattice $\Lambda \in \mathbb{R}^{N-1}$.

Theorem 1. Let $\Lambda_{\mathbf{e}} = \mathbb{Z} \oplus c_2\mathbb{Z} \dots \oplus c_N\mathbb{Z}$, $c_i \in \mathbb{R}$. Let Λ be a target lattice in \mathbb{R}^{N-1} and consider a lower triangular generator matrix $L^* = (l_{ij}^*)$ for Λ^* . If Λ_w^* , $w \in \mathbb{N}$ is the sequence of lattices generated by the matrices

$$L_w^* := \begin{pmatrix} \lfloor wl_{11}^* \rfloor & \frac{1}{c_2} & \dots & \dots & 0 \\ \lfloor wl_{21}^* \rfloor & \frac{\lfloor wl_{22}^* c_2 \rfloor}{c_2} & \dots & \dots & 0 \\ \vdots & \vdots & \ddots & \dots & 0 \\ \lfloor wl_{n1}^* \rfloor & \frac{\lfloor wl_{n2}^* c_2 \rfloor}{c_2} & \dots & \frac{\lfloor c_{n-1} wl_{nn}^* \rfloor}{c_{n-1}} & \frac{1}{c_n} \end{pmatrix}, \quad (19)$$

then:

(i) $L_w^* = P_{\hat{\mathbf{u}}^\perp}(\Lambda)^*$ for some $\hat{\mathbf{u}} \in \mathbb{R}^N$ and

(ii) $(1/w^2)L_w^*L_w^{*t} \rightarrow L^*L^{*t}$ as $w \rightarrow \infty$.

In other words, for large w , Λ will be approximated (in the sense of the Gram matrices) by projections of $\Lambda_{\mathbf{c}}$.

Proof: Through applying elementary (integer) operations on L_w^* we can put it on form (18) for some integers u_2, \dots, u_n depending on w , hence L_w^* is a generator matrix for $P_{\hat{\mathbf{u}}^\perp}(\Lambda)^*$, proving the first statement. For the second statement, we clearly have $(1/w)L_w^* \rightarrow [L^* \ \mathbf{0}]$ as $w \rightarrow \infty$, where $\mathbf{0}$ is the $(n-1) \times n$ all-zero column vector. Therefore, $(1/w^2)L_w^*L_w^{*t} \rightarrow [L^* \ \mathbf{0}][L^* \ \mathbf{0}]^t = L^*L^{*t}$. ■

Since the density of a lattice is a continuous function of the entries of its Gram matrix, it follows that the sequence of curves produced by the vectors $\hat{\mathbf{u}}$ approaches the bound (14) as $w \rightarrow \infty$.

Example 3. Consider the hexagonal lattice [2], which is the best packing in two dimensions and is equivalent to its dual. One of its generator matrix is

$$L^* = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}.$$

We apply the construction above and reduce, through elementary operations, the matrix L_w^* in order to put it on form (18). After re-scaling the rectangular lattice $\Lambda_{\mathbf{c}}$, we find the sequence of vectors

$$\hat{\mathbf{u}}_w = (c_1, -2wc_2, (2w\lfloor w\sqrt{3}c_2/c_1 \rfloor - w)c_3).$$

The projections of $c_1\mathbb{Z} \oplus c_2\mathbb{Z} \oplus c_3\mathbb{Z}$ onto $\hat{\mathbf{u}}_w^\perp$ will be, up to equivalence, arbitrarily close to A_2 when $w \rightarrow \infty$.

B. Comparisons: Curves in \mathbb{R}^6

Here we compare our approach of constructing curves on torus layers with the curves constructed in [7] and [9] in terms of length for given small-ball radii. Given $\delta > 0$, we first consider a set of flat tori associated to a spherical code in \mathbb{R}^3 , with minimum distance greater than 2δ , as described in Section IV-A. Through the first inequality of (13), for each torus, we can find $r_{\mathbf{c}}$ in order to guarantee that the curves on the flat tori will have small-ball radius at least δ (this is also done in the case of the curves on the torus $c = \hat{\mathbf{e}}$ used in [9]). We then look for the larger element of the sequence of vectors that produces a projection with minimum distance at least $r_{\mathbf{c}}$.

In Figure 4, the first curve from the bottom to the top represents the exponential sequence [9]. The second one is obtained by directly applying the Lifting Construction [7] to the hexagonal lattice and the last one displays the total length associated to our scheme.

VI. MEAN SQUARE ERROR ANALYSIS

The first step of the encoding process described in Section IV-C involves a choice of partition of the interval $[0, 1]$. If the source to be transmitted is uniformly distributed over $[0, 1]$, it is intuitive that the best choice in terms

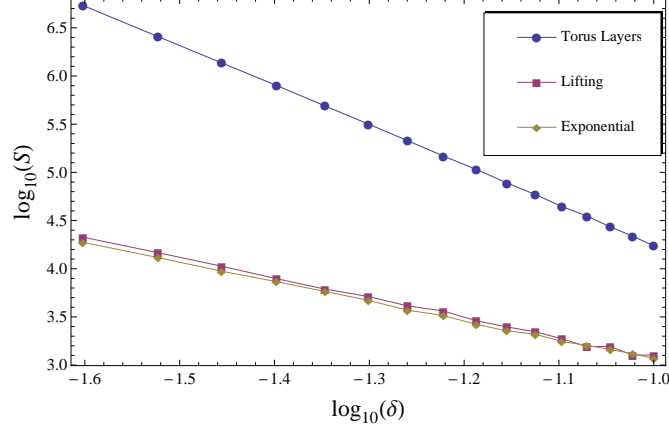


Fig. 4: Comparison between diferent approaches in terms of small-ball radius δ and arc-length \mathcal{S} .

of mse is the one such that the encoding map will have constant stretch. This fact is formalized in the next theorem and holds for uniform sources.

Theorem 2. *Let I_1, \dots, I_M be a partition of the interval $[0, 1]$. Let $\{\mathbf{s}_1, \dots, \mathbf{s}_M\}$ be a collection of non-overlapping curves on \mathcal{S}^{2N-1} of length L_1, \dots, L_M and with domain I_1, \dots, I_M respectively. Suppose the encoding rule*

$$\mathbf{s}(x) = \mathbf{s}_j(x) \text{ if } x \in I_j$$

is used. Then, under the low noise regime, the minimum ϵ^ of $E_{\text{low}}[(X - \hat{X})^2]$ over all partitions $\{I_1, \dots, I_m\}$ is achieved by choosing I_j such that*

$$|I_j| = \frac{L_j}{\sum_{j=1}^M L_j} = \frac{L_j}{L} \quad (20)$$

and satisfies $\epsilon^ = \sigma^2/L^2$.*

Proof: First note that

$$\begin{aligned} E_{\text{low}}[(X - \hat{X})^2] &= \sum_{j=1}^M E_{\text{low}}[(X - \hat{X})^2 | X \in I_j] P(X \in I_j) \\ &= \sum_{j=1}^M E_{\text{low}}[(X - \hat{X})^2 | X \in I_j] |I_j| \end{aligned}$$

and if the noise is small, $E_{\text{low}}[(X - \hat{X})^2 | X \in I_j] = \sigma^2 |I_j|^2 / L_j^2$. Therefore

$$E_{\text{low}}[(X - \hat{X})^2] = \sigma^2 \sum_{j=1}^M \frac{|I_j|^3}{L_j^2} = \sigma^2 L \sum_{j=1}^M \left(\frac{|I_j|}{L_j} \right)^3 \left(\frac{L_j}{L} \right)$$

$$\sigma^2 \stackrel{(a)}{\geq} L \sum_{j=1}^M \left(\frac{|I_j|}{L} \right)^3 = \frac{\sigma^2}{L^2},$$

where the inequality (a) is due to the strict convexity of the function $g(x) = x^3$ for $x > 0$. Equality in (a) only holds when $|I_j|/L_j = |I_k|/L_k, \forall k, j$. This condition, together with $\sum_{j=1}^n |I_j| = 1$, implies Equation (20).

217

It was proved in [9] that for fixed N and σ^2 , the mse decays with order $O(1/P^N)$ provided that the parameter a [9, Sec. VI. A] is chosen suitably (i.e., the mse decays non-linearly with the power). In what follows we prove that the lengths of the curves constructed from the torus layer scheme are about $O(N!)$ greater than the scheme proposed in [9], even if we use the improvements provided by the Lifting Construction [7]. This will yield an asymptotic behavior comparable to the $O(1/P^N)$ in [9], but with a better performance when N increases.

Let N be fixed and let $\mathbf{c}(t)$ be given by Equation (8). Recall that $\hat{\mathbf{e}} = 1/\sqrt{N}(1, \dots, 1)$. For $\rho > 0$ sufficiently small, there is t such that the tori in the set $\mathcal{SC}_{\mathbf{c}(t)}$ have minimum distance 2ρ . Moreover, there is $\hat{\mathbf{u}} \in \Lambda_{\mathbf{c}(t)}$ arbitrarily close to the bound (14) i.e., such that the curve in each torus has length:

$$L_{\mathbf{c}(t)} = 2\pi \|\hat{\mathbf{u}}\| = \frac{2^{N-1}}{\rho^N} \prod_{i=1}^N c_i(t) \left(\frac{\Delta_{N-1}}{\mathcal{V}_{N-1}} - \varepsilon_1 \right).$$

The same vector \mathbf{u} can be used for any $\mathbf{c} \in \mathcal{SC}_{\mathbf{c}(t)}$ just by interchanging its coordinates, yielding the same parameters for all tori.

Since we have $N!$ tori in the set $\mathcal{SC}_{\mathbf{c}(t)}$, the total length L_{TL} produced is given by:

$$L_{TL} = (N!)L_{\mathbf{c}(t)} = N! \frac{2^{N-1}}{\rho^N} \prod_{i=1}^N c_i(t) \left(\frac{\Delta_{N-1}}{\mathcal{V}_{N-1}} - \varepsilon_1 \right)$$

For the Lifting Construction and the same small-ball radius, we have

$$L_{LC} = L_{\hat{\mathbf{e}}} = 2\pi \|\hat{\mathbf{u}}\| = \frac{2^{N-1}}{\rho^N N^{N/2}} \left(\frac{\Delta_{N-1}}{\mathcal{V}_{N-1}} - \varepsilon_2 \right).$$

As $\rho \rightarrow 0$, we can make both ε_1 and ε_2 vanish, and then

$$\frac{L_{TL}}{L_{LC}} \rightarrow N! N^{N/2} \prod_{i=1}^N c_i(t).$$

Now, by making t sufficiently small (but keeping the small-ball radius greater than ρ), we have

$$\frac{L_{TL}}{L_{LC}} \rightarrow N! \text{ as } t \rightarrow 0.$$

In other words,

231

$$E_{\text{low}}^{TL}[(X - \hat{X})^2] \approx \frac{1}{(N!)^2} E_{\text{low}}^{LC}[(X - \hat{X})^2],$$

where the approximation is good when we allow the small-ball radius to be sufficiently small (this is the case when the SNR is high).

VII. SIMULATIONS

We now present some simulation results in order to illustrate the performance of our proposed scheme. In all simulations we consider a source uniformly distributed over the interval $[-1/2, 1/2]$.

As a benchmark we chose the spherical code scheme (V&C) presented in [9], since this scheme outperforms the one in [10] for uniform sources, as shown in their simulations. We also plotted the curves corresponding to a simple linear modulation.

The simulation results corresponds to 50000 samples with 1 to 6 expansion rate ($N = 3$) and the performance is measured in terms of $1/\text{mse}$ as a function of the $\text{SNR} = P/\sigma^2$.

In Figure 5 it is displayed the torus layer scheme performance for $M = 6$ tori designed as in Example 1 with $t = 0.6$, and projection vector $\mathbf{u} = (1, 2, 198)$. The spherical code scheme V&C was simulated for $a = 18$. The parameters were chosen such that both schemes reach the asymptote for the same SNR (i.e., have approximately the same small-ball radius). As expected, our scheme outperforms (V&C) in the low noise regime (about 13dB higher in this example).

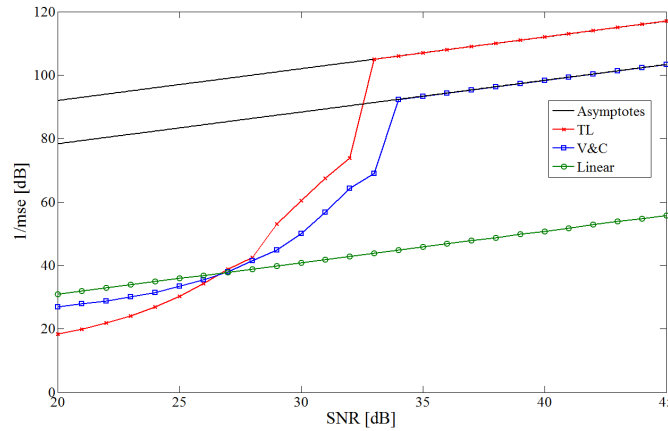


Fig. 5: Both schemes reach their asymptote for the same SNR. The torus layer scheme outperforms (V&C) in the low noise regime.

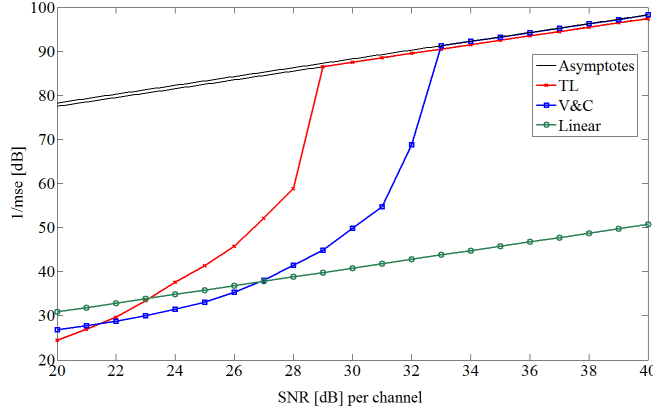


Fig. 6: Both schemes have about the same performance in low noise regime. Torus layers reaches the asymptote for a smaller SNR.

Figure 6 shows the simulations results for $M = 6$ tori, also designed as in Example 1 with $t = 1.75$, and projection vector $\mathbf{u} = (1, 4, 34)$. The spherical code scheme V&C was simulated also for $a = 18$. These parameters were chosen such that both schemes achieve about the same performance in low noise regime (same asymptote). In this case, our scheme reaches the asymptote for a smaller SNR (5dB in this example). In both graphics it is possible to see that the linear modulation is a good choice only for very small SNR.

We finish this section with a remark on the implementation. Since the curves constructed from our method are closed, if the sent value is one extreme value of the interval $I_j = [L_{j-1}/L, L_j/L]$ it may happen that the decoder wrongly interpret it as the other extreme. In other words, if $L_{j-1}/L + \varepsilon$ is sent, then with a non-negligible probability, the decoder will output L_j/L . To prevent these errors, the encoder needs to “shrink” the interval I_j by a factor $\alpha \in (0, 1)$, such that the curve is “opened”, separating the encoded extremes values apart. However, this process deteriorates the low-noise performance, so that α needs to be calibrated. In our examples, it sufficed to take any α between 0.7 and 0.8. This issue was briefly commented in [9].

VIII. CONCLUSION

The problem of transmitting a continuous alphabet source over an AWGN channel was considered through an approach based on curves designed in layers of flat tori on the surface of a $(2N)$ -dimensional Euclidean sphere. This approach explores connections with constructions of spherical codes and is related to the problem of finding dense projections of the lattice $c_1\mathbb{Z} \oplus \dots \oplus c_N\mathbb{Z}$.

This work is a generalization of both the scheme proposed in [9] and the Lifting Construction in [7]. As a consequence, our scheme compares favorably to previous works in terms of the tradeoff between total length and small-ball radius, which is a proper figure of merit for this communication system.

In spite of the improvements in terms of length versus small-ball radius, the constructiveness, homogeneity and overall complexity of the decoding algorithm are features preserved from [9]. Simulations also show a good scaling

of the mse with the channel SNR. Further questions include applications to sources that do not have limited support, or when it is not possible to employ a companding function (e.g., Gaussian mixtures), and generalization to other bandwidth expansion ratios then $1 : 2N$ (the general case is to consider K to N expansion mappings, when $1 < K < N$).

ACKNOWLEDGMENT

The authors would like to thank the Centre Interfacultaire Bernoulli (CIB) at EPFL where part of this work was developed, during the special semester on Combinatorial, Algebraic and Algorithmic Aspects of Coding Theory. The authors also thank Vinay Vaishampayan for helpful discussions on analog source-channel coding.

REFERENCES

- [1] M Berger and S. Gostiaux. *Differential Geometry: Manifolds, Curves and Surfaces*. Berlin: Springer-Verlag, 1998.
- [2] J. H. Conway and N. J. A. Sloane. *Sphere-packings, lattices, and groups*. Springer-Verlag, New York, NY, USA, 1998.
- [3] T Ericson and V Zinoviev. *Codes on Euclidean Spheres*. North-Holland Mathematical Library, 2001.
- [4] J. Hamkins and K. Zeger. Asymptotically dense spherical codes. i. wrapped spherical codes. *IEEE Transactions on Information Theory*, 43(6):1774–1785, Nov. 1997.
- [5] J. Martinet. *Perfect Lattices in Euclidean Space*. Springer-Verlag, Berlin Heidelberg New York, 2003.
- [6] D. J. Sakrison. *Notes on analog communications*. New York: Van Nostrand Reinhold, 1970.
- [7] N. J. A. Sloane, V. A. Vaishampayan, and S. I. R. Costa. The lifting construction: A general solution for the fat strut problem. In *IEEE International Symposium on Information Theory Proceedings (ISIT)*, pages 1037 –1041, 2010.
- [8] C. Torezzan, S. I. R Costa, and V. Vaishampayan. Spherical codes on torus layers. In *IEEE International Symposium on Information Theory Proceedings (ISIT)*, pages 2033–2037, 2009.
- [9] V. A. Vaishampayan and S. I. R. Costa. Curves on a sphere, shift-map dynamics, and error control for continuous alphabet sources. *IEEE Transactions on Information Theory*, 49:1658–1672, 2003.
- [10] N. Wernersson, M. Skoglund, and T. Ramstad. Polynomial based analog source-channel codes. *IEEE Transactions on Communications*, 57(9):2600 –2606, september 2009.